A model structure on relative dg-Lie algebroids

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Abstract

In this Note, for the future purposes of relative formal derived deformation theory and of derived coisotropic structures, we prove the existence of a model structure on the category of dg-Lie algebroids over a cochain differential non-positively graded commutative algebra over a commutative base $\mathbb{Q}$-algebra k.

Notations.

- k will denote the base ring and will be assumed to be of characteristic 0.
- $\mathsf{C}(k)$ will denote the model category of unbounded complexes of k-modules with surjections as fibrations, and quasi-isomorphisms s equivalences. It is a symmetric monoidal model category with the ususal tensor product $\otimes_k$ of complexes over k.
- cdga_k denotes the category of differential *nonpositively* graded algebras over k, with differential increasing the degree by 1. We will always consider cdga_k endowed with the usual model structure for which fibrations are surjections in negative degrees, and equivalences are quasi-isomorphisms (see [HAGII, §2.2.1]).
- A dg-Lie algebra over k is a Lie algebra object in the symmetric monoidal category $(\mathsf{C}(k), \otimes_k)$, i.e. an object $\mathfrak{g}$ in $\mathsf{C}(k)$ endowed with a morphism of complexes (called the bracket) $[-,-] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ such that
  - $[x, y] = (-1)^{|x||y|+1}[y, x]$
  - $[x, [y, z]] + (-1)^{(|z||x|+|x||y|)}[y, [z, x]] + (-1)^{(|z||x|+|z||y|)}[z, [y, x]] = 0$
  - $d[x, y] = [dx, y] + (-1)^{|x|}[x, dy]$

  A morphism of dg-Lie algebras over k is a morphism of complexes commuting with the brackets.

- We'll write $\mathsf{dgLieAlg}_k$ for the (left proper, combinatorial) model category of unbounded dg-Lie algebras over k, where fibrations and equivalences are defined via the forgetful functor $\mathsf{dgLieAlg}_k \to \mathsf{C}(k)$. $\mathsf{dgLieAlg}_k$ is a simplicial model category via $\underline{\mathrm{Hom}}(\mathfrak{g}, \mathfrak{h})_n := \mathrm{Hom}_{\mathsf{dgLieAlg}_k}(\mathfrak{g}, \Omega^\bullet(\Delta_n) \otimes_k \mathfrak{h})$. The associated ∞-category (i.e. the Dwyer-Kan localization of $\mathsf{dgLieAlg}_k$ along quasi-isomorphisms) will be denoted by $\mathbf{dgLieAlg}_k$.
- By ∞-category, we mean either a simplicial or a quasi-category ([Lu-HTT]).

1 Relative dg-Lie algebroids

We denote by $(Q : \mathsf{cdga}_k \longrightarrow \mathsf{cdga}_k, Q \to \mathrm{Id})$ a cofibrant replacement functor in the model category cdga_k together with the associated natural transformation. For $A \in \mathsf{cdga}_k$, we will write T_A for the tangent dg-Lie algebra of self-derivations of A. Recall that there is a natural k-Lie bracket on $A \oplus \mathrm{T}_A$ ([B-B]).

Definition 1.1
- *Let $A \in \mathsf{cdga}_k$ be cofibrant. The category $\mathsf{dgLieAlgd}_A$ of* dg-Lie algebroids over *A is the category whose*
  - objects *are the pairs $(L, \alpha : L \to T_A)$ where L is a A-dg-module and a k-dg-Lie algebra, and α (called the anchor map) is a morphism of A-dg-modules and of k-dg-Lie algebras, such that for any homogeneous $\ell_1, \ell_2 \in L$ and any homogeneous $a \in A$, the following* graded Leibniz rule *holds*
  $$[\ell_1, a\ell_2] = (-1)^{|a||\ell_1|} a[\ell_1, \ell_2] + \alpha(\ell_1)(a)\ell_2$$
  - *as* morphisms *$(L', \alpha') \to (L, \alpha)$ morphisms $\psi : L' \to L$ of A-dg-modules and of k-dg-Lie algebras, commuting with anchor maps. A morphism in $\mathsf{dgLieAlgd}_A$ is an* equivalence *if it is a quasi-isomorphism*
- *For any $A \in \mathsf{cdga}_k$, the category $\mathsf{ddgLieAlgd}_A$ of* derived dg-Lie algebroids over *A is the category $\mathsf{dgLieAlgd}_{QA}$.*
- *For any $A \in \mathsf{cdga}_k$, the ∞-category $\mathbf{dgLieAlgd}_A$ of* derived dg-Lie algebroids over *A is the Dwyer-Kan localization of $\mathsf{ddgLieAlgd}_{QA}$ with respect to equivalences.*

Examples. When $A = k$ we get the category of dg-Lie k-algebras.

Theorem 1.2 *Let $A \in \mathsf{cdga}_k$ be cofibrant. Then the category $\mathsf{dgLieAlgd}_A$ of dg-Lie algebroids over A, endowed with equivalences and fibrations defined on the underlying A-dg-modules, is a cofibrantly generated model category.*

Proof. We will transfer the model structure from $A - \mathsf{dgMod}/T_A$ to $\mathsf{dgLieAlgd}_A$ via the adjunction

$$A - \mathsf{dgMod}/T_A \underset{\text{Forget}}{\overset{\text{Free}}{\rightleftarrows}} \mathsf{dgLieAlgd}_A$$

where the Free functor, i.e. the left adjoint, is defined in [Ka]. In order to do this, since $A-\mathsf{dgMod}/T_A$ is a cofibrantly generated model category and $\mathsf{dgLieAlgd}_A$ has small colimits and finite limits, we may use the following transfer criterion, originally due to Quillen, (see e.g. [B-M, 2.5, 2.6]).

Transfer criterion. Define a map f in $\mathsf{dgLieAlgd}_A$ to be a weak equivalence (resp. a fibration) if Forget(f) is a weak equivalence (resp. a fibration). This defines a cofibrantly generated model structure on $\mathsf{dgLieAlgd}_A$, provided

(i) the functor Forget preserves small objects;
(ii) $\mathsf{dgLieAlgd}_A$ has a fibrant replacement functor, and functorial path-objects for fibrant objects.

Now, condition (i) follows easily from the fact that the forgetful functor commutes with filtered colimits. For the first part of condition (ii) we may take the identity functor as a fibrant replacement functor, since all objects in $\mathsf{dgLieAlgd}_A$ are fibrant. To prove the second half of condition (ii) - i.e. the existence of functorial path-objects - we consider the following construction. Recall first that if $L_1 \to T_A$ and $L_2 \to T_A$ are dg-Lie algebroids over A, then their product in $\mathsf{dgLieAlgd}_A$ is $L_1 \times_{T_A} L_2 \to T_A$, with the obvious induced k-dg-Lie bracket.
Let now $(\alpha : L \to T_A) \in \mathsf{dgLieAlgd}$, and consider the A-dg-module $L[t, dt] := L \otimes_k k[t, dt]$, where $k[t, dt] := k[t] \oplus k[t]dt$ is identified with the de Rham commutative dg-algebra of $\mathrm{Spec}(k[x_0, x_1]/(x_0 + x_1 - 1))$ with $k[t]$ sitting in degree 0, and $k[t]dt$ in degree 1, and the differential is given by

$$d(\ell_1 f(t) + \ell_2 g(t)dt) := d_L(\ell_1)f(t) + ((-1)^{|\ell_1|}\ell_1 f'(t) + d_L(\ell_2)g(t))dt.$$

We endow $L[t, dt]$ with the following k-dg-Lie bracket

$$[\ell_1 f_1(t), \ell_2 f_2(t)] := [\ell_1, \ell_2]_L f_1(t)f_2(t), \qquad [\ell_1 f_1(t), \ell_2 g_2(t)dt] := [\ell_1, \ell_2]_L f_1(t)g_2(t)dt.$$

Note that for any $s \in k$, we have an evaluation map

$$\mathrm{ev}_s : L[t, dt] \longrightarrow L\,, \quad \ell_1 f(t) + \ell_2 g(t)dt \longmapsto f(s)\ell_1$$

that is a morphism of A-dg-modules and of k-dg-Lie algebras, and a quasi-isomorphism.
Consider now the pull-back diagram in $A - \mathsf{dgMod}$

$$\begin{tikzcd} \mathrm{Path}(L \to T_A) \arrow[r] \arrow[d] & L[t, dt] \arrow[d] \\ T_A \arrow[r] & T_A[t, dt] \end{tikzcd} ,$$

where the right vertical map is given by

$$L[t, dt] \longrightarrow T_A[t, dt]\,, \quad \sum_i \xi_i f_i(t) + \eta_i g_i(t)dt \longmapsto \sum_i \alpha(\xi_i)f_i(t) + \alpha(\eta_i)g_i(t)dt,$$

while the map $T_A \to T_A[t, dt]$ is the obvious inclusion $\xi \mapsto \xi\cdot 1 + 0\cdot dt$. In other words, $\mathrm{Path}(\, L \xrightarrow{\alpha} T_A\,)$ is the sub-A-dg module of $L[t, dt]$ consisting of elements $\sum_i \xi_i f_i(t) + \eta_i g_i(t)dt$ such that

$$\sum_i \alpha(\xi_i)f_i(t) \in T_A \hookrightarrow T_A[t]\,, \quad \sum_i \alpha(\eta_i)g_i(t)dt = 0 \text{ in } T_A \hookrightarrow T_A[t]dt.$$

It is straightforward that the composition

$$\mathrm{Path}(L \to T_A) \longrightarrow L[t,dt] \xrightarrow{(\mathrm{ev}_0,\mathrm{ev}_1)} L \times L$$

factors through the inclusion $L \times_{T_A} L \hookrightarrow L \times L$, and that the resulting diagram

$$\begin{tikzcd} \mathrm{Path}(L \to T_A) \arrow[rr, "p"] \arrow[dr] & & L \times_{T_A} L \arrow[dl] \\ & T_A & \end{tikzcd}$$

commutes. Moreover, $\mathrm{Path}(\, L \xrightarrow{\alpha} T_A \,)$ is closed under the k-dg-Lie bracket in $L[t,dt]$, and the canonical map $\mathrm{Path}(\, L \xrightarrow{\alpha} T_A \,) \to T_A$ is a morphism of k-dg-Lie algebras. The fact that $(\mathrm{Path}(\, L \xrightarrow{\alpha} T_A \,) \to T_A)$ is actually a dg-Lie algebroid over A, i.e. satisfies also the graded Leibniz rule, follows from a direct computation. Namely, the lhs of the Leibniz identity is

$$[\sum_i \xi_i f_i(t) + \eta_i g_i(t)dt, a(\sum_j \overline{\xi}_j \overline{f}_j(t) + \overline{\eta}_j \overline{g}_j(t)dt)] =$$

$$= \sum_{i,j} (-1)^{|a||\xi_i|} a[\xi_i, \overline{\xi}_j] f_i(t) \overline{f}_j(t) + (-1)^{|a||\xi_i|} a[\xi_i, \overline{\eta}_j] f_i(t) \overline{g}_j(t)dt + (-1)^{|a||\eta_i| + (|a| + |\overline{\xi}_j|)} a[\eta_i, \overline{\xi}_j] \overline{f}_j(t) g_i(t)dt +$$

$$+ \alpha(\xi_i)(a) \overline{\xi}_j f_i(t) \overline{f}_j(t) + \alpha(\xi_i)(a) \overline{\eta}_j f_i(t) \overline{g}_j(t)dt + (-1)^{|a| + |\overline{\xi}_j|} \alpha(\eta_i)(a) \overline{\xi}_j \overline{f}_j(t) g_i(t)dt;$$

we observe that, since $(\sum_i \xi_i f_i(t) + \eta_i g_i(t)dt) \in \mathrm{Path}(\, L \xrightarrow{\alpha} T_A \,)$, the last term $\alpha(\eta_i)(a)\overline{\xi}_j \overline{f}_j(t) g_i(t)dt$ vanishes, and moreover $\sum_i \alpha(\xi_i) f_i(t) + \alpha(\eta_i) g_i(t)dt = \partial$, for some $\partial \in T_A$, so that

$$[\sum_i \xi_i f_i(t) + \eta_i g_i(t)dt, a(\sum_j \overline{\xi}_j \overline{f}_j(t) + \overline{\eta}_j \overline{g}_j(t)dt)] =$$

$$= \sum_{i,j} ((-1)^{|a||\xi_i|} a[\xi_i, \overline{\xi}_j] f_i(t) \overline{f}_j(t) + (-1)^{|a||\xi_i|} a[\xi_i, \overline{\eta}_j] f_i(t) \overline{g}_j(t)dt + (-1)^{|a||\eta_i| + (|a| + |\overline{\xi}_j|)} a[\eta_i, \overline{\xi}_j] \overline{f}_j(t) g_i(t)dt) +$$

$$+ \sum_j \partial(a) \overline{\xi}_j \overline{f}_j(t) + \partial(a) \overline{\eta}_j \overline{g}_j(t)dt.$$

The rhs of the Leibniz identity reads

$$\sum_{i,j} [\xi_i f_i(t), a(\sum_j \overline{\xi}_j \overline{f}_j(t) + \overline{\eta}_j \overline{g}_j(t)dt)] + [\eta_i g_i(t)dt, a(\sum_j \overline{\xi}_j \overline{f}_j(t) + \overline{\eta}_j \overline{g}_j(t)dt))] =$$

$$= \sum_{i,j} ((-1)^{|a||\xi_i|} a[\xi_i f_i(t), \overline{\xi}_j \overline{f}_j(t) + \overline{\eta}_j \overline{g}_j(t)dt)] + (-1)^{|a|(|\eta_i+1|} a[\eta_i g_i(t)dt, \overline{\xi}_j \overline{f}_j(t) + \overline{\eta}_j \overline{g}_j(t)dt)]) +$$

$$+ \partial(a)(\sum_j \overline{\xi}_j \overline{f}_j(t) + \overline{\eta}_j \overline{g}_j(t)dt) =$$

$$= \sum_{i,j} ((-1)^{|a||\xi_i|} a[\xi_i, \overline{\xi}_j] f_i(t) \overline{f}_j(t) + (-1)^{|a||\xi_i|} a[\xi_i, \overline{\eta}_j] f_i(t) \overline{g}_j(t) dt + (-1)^{|a||\eta_i| + (|a| + |\overline{\xi}_j|)} a[\eta_i, \overline{\xi}_j] \overline{f}_j(t) g_i(t) dt) +$$

$$+ \sum_j \partial(a) \overline{\xi}_j \overline{f}_j(t) + \partial(a) \overline{\eta}_j \overline{g}_j(t) dt,$$

and we conclude that $(\mathrm{Path}(\ L \xrightarrow{\alpha} T_A\ ) \to T_A)$ is indeed a dg-Lie algebroid over A.

Finally, the diagonal map of dg-Lie algebroids over A

$$L \longrightarrow L \times_{T_A} L$$

factors as

$$L \xrightarrow{u} \mathrm{Path}(L \to T_A) \xrightarrow{p} L \times_{T_A} L\ ,$$

where $u(\xi) = \xi \cdot 1 + 0 \cdot dt$. Now, p is surjective, since for any $(\xi, \eta) \in L \times_{T_A} L$, we have that $\xi t + \eta(1-t) \in \mathrm{Path}(L \to T_A)$ and $p(\xi t + \eta(1-t)) = (\xi, \eta)$; hence p is a fibration. The following Lemma concludes the proof of the existence of functorial path-objects in $\mathsf{dgLieAlgd}_A$ (hence of the existence of a transferred model structure on $\mathsf{dgLieAlgd}_A$).

Lemma 1.3 *The map $u : L \longrightarrow \mathrm{Path}(L \to T_A)$ is a weak equivalence (i.e. a quasi-isomorphism).*

Proof of lemma. First of all, we notice that the evaluation-at-0 map

$$\mathrm{ev}_0 : \mathrm{Path}(L \to T_A) \longrightarrow L\ , \ \sum_i \xi_i f_i(t) + \eta_i g_i(t) dt \longmapsto \sum_i \xi_i f_i(0)$$

is a left inverse to u. So it will be enough to produce a homotopy between $u \circ \mathrm{ev}_0$ and $\mathrm{Id}_{\mathrm{Path}(L \to T_A)}$. Let us define a family of k-linear maps, indexed by $p \in \mathbb{Z}$,

$$h^p : \mathrm{Path}(L \to T_A)^p \longrightarrow \mathrm{Path}(L \to T_A)^{p-1}\ , \quad \xi f(t) + \eta g(t) dt \longmapsto (-1)^p \eta \int_0^t g(x) dx.$$

We pause a moment to explain why this maps are well defined. We have used that if $\theta := \sum_i \xi_i f_i(t) + \eta_i g_i(t) dt$ is homogeneous of degree p (so that $|\xi_i| = p$ and $|\eta_i| = p - 1$, for all i), then $h^p(\theta)$, which a priori belongs to $L[t, dt]^{p-1}$, actually belongs to $\mathrm{Path}(L \to T_A)^{p-1}$. This can be seen as follows. Write $\sum_i \eta_i g_i(t) dt$ as $\sum_{s \geq 0} \tilde{\eta}_s t^s dt$, where each $\tilde{\eta}_s$ is a k-linear combination of all the η_i's. Then the condition that $\theta \in \mathrm{Path}(L \to T_A)^p$ reads $\sum_{s \geq 0} \alpha(\tilde{\eta}_s) t^s dt = 0$, i.e. that $\alpha(\tilde{\eta}_s) = 0$ for all s. Therefore

$$h^p(\theta) = \sum_{s \geq 0} \tilde{\eta}_s t^{s+1} / (s+1) dt$$

belongs to $\mathrm{Path}(L \to T_A)^{p-1}$, as claimed.
Now, it is a straightforward verification that the family of maps $\{h^p\}$ yields a homotopy between $u \circ \mathrm{ev}_0$ and $\mathrm{Id}_{\mathrm{Path}(L \to T_A)}$, i.e. that

$$(h^{p+1} \circ d + d \circ h^p)(\xi f(t) + \eta g(t) dt) = -\xi(f(t) - f(0)) - \eta g(t) dt = (u \circ \mathrm{ev}_0 - \mathrm{Id}_{\mathrm{Path}(L \to T_A)})(\xi f(t) + \eta g(t) dt).$$

$\square$ $\square$

Remark 1.4 Exactly the same argument proves that we may transfer the *tame* model structure from $A-\mathsf{dgMod}/T_A$ to $\mathsf{dgLieAlgd}_A$ via the same adjunction. In fact, fibrations remain the same, and Lemma 1.3 is still true since a homotopy equivalence is a tame equivalence.

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